Lecture 9
The Greedy Method

CS 161 Design and Analysis of Algorithms

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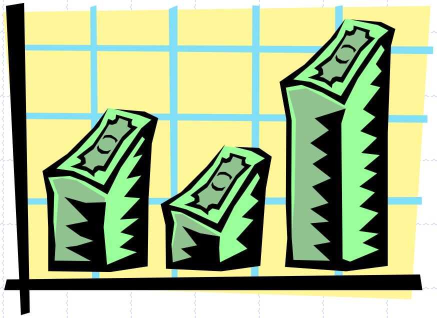
Application: Web Auctions

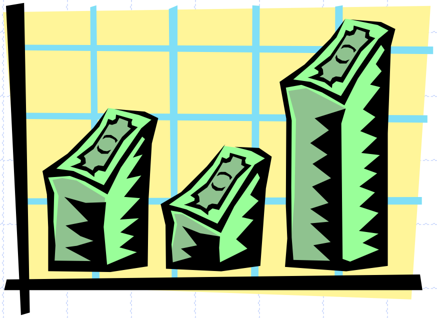
- Suppose you are designing a new **online auction website** that is intended to process bids for multi-lot auctions.
- This website should be able to handle a single auction for 100 units of the same digital camera or 500 units of the same smartphone, where bids are of the form, **"x units for $y,"** meaning that the bidder wants a quantity of x of the items being sold and is willing to pay $y for all x of them.
- The challenge for your website is that it must allow for a large number of bidders to place such multi-lot bids and it must decide which bidders to choose as the winners.
- Naturally, one is interested in designing the website so that it always chooses a set of winning bids that maximizes the total amount of money paid for the items being auctioned.
- So how do you decide which bidders to choose as the winners?

The Greedy Method

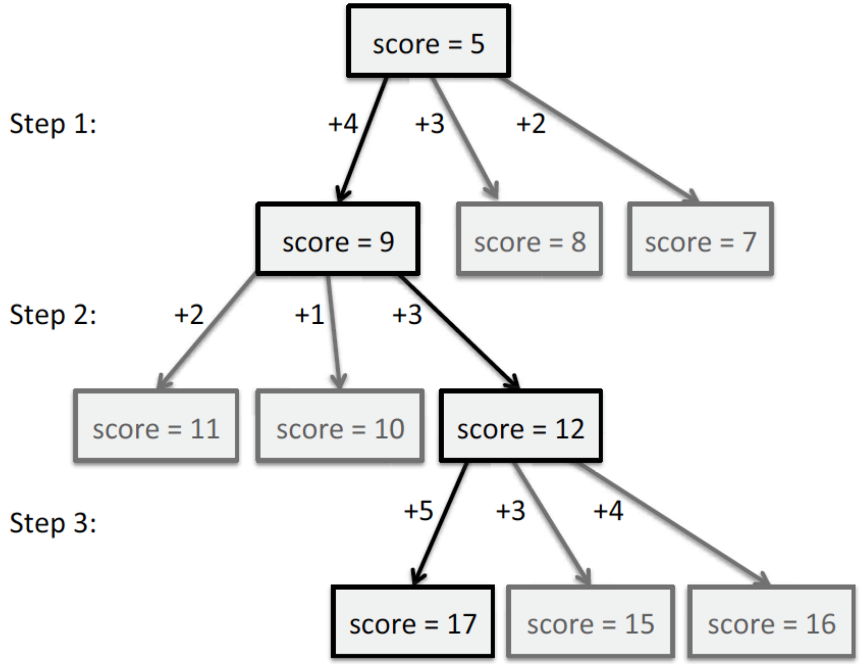


- **The greedy method** is a general algorithm design paradigm, built on the following elements:
  - **configurations**: different choices, collections, or values to find
  - **objective function**: a score assigned to configurations, which we want to either maximize or minimize
- It works best when applied to problems with the **greedy-choice** property:
  - a globally-optimal solution can always be found by a series of local improvements from a starting configuration.

The Greedy Method



- The sequence of choices starts from some well-understood starting configuration, and then iteratively makes the decision that is best from all of those that are currently possible, in terms of improving the objective function.



Web Auction Application

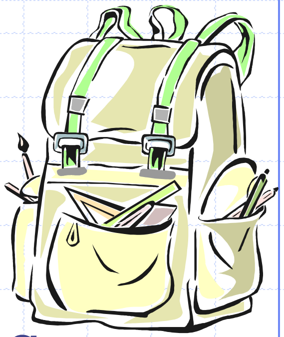
- This greedy strategy works for the profit-maximizing online auction problem if you can satisfy a bid to buy x units for $y by selling k < x units for $k * y/x.
- In this case, this problem is equivalent to the **fractional knapsack problem**.

American GIs recover works of art stolen by the Nazis (NARA/Public Domain)

Web Auctions and the Fractional Knapsack Problem

- In the **knapsack problem**, we are given a set of n items, each having a weight and a benefit, and we are interested in choosing the set of items that maximize our total benefit while not going over the weight capacity of the knapsack.
- In the web auction application, each bid is an item, with its "weight" being the number of units being requested and its benefit being the amount of money being offered.
- In the instance, where bids can be satisfied with a partial fulfillment, then it is an instance of the **fractional** knapsack problem, for which the greedy method works to find an optimal solution.
- Interestingly, for the "0-1" version of the problem, where fractional choices are not allowed, then the greedy method may not work and the problem is potentially very difficult to solve in polynomial time.

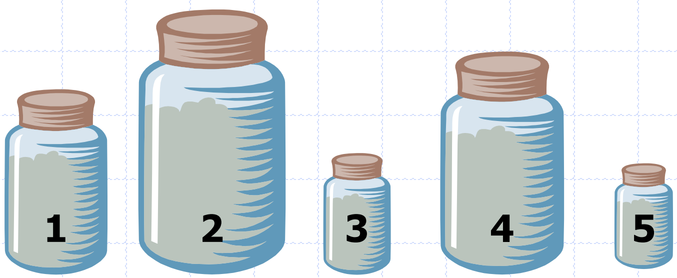
The Fractional Knapsack Problem



- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are allowed to take fractional amounts, then this is the **fractional knapsack problem**.
  - In this case, we let x_i denote the amount we take of item i
  - Objective: maximize $$\sum_{i \in S} b_i (x_i / w_i)$$
  - Constraint: $$\sum_{i \in S} x_i \le W$$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.



Items:	1	2	3	4	5
Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	\$12	\$32	\$40	\$30	\$50
Value: ($ per ml)	3	4	20	5	50

“knapsack”

10 ml

Solution:
- 1 ml of 5
- 2 ml of 3
- 6 ml of 4
- 1 ml of 2

The Fractional Knapsack Algorithm

- Greedy choice: Keep taking item with highest value (benefit to weight ratio)
  - Since $\sum_{i \in S} b_i (x_i / w_i) = \sum_{i \in S} (b_i / w_i) x_i$
  - Run time: O(n log n). Why?
- Correctness: Suppose there is a better solution
  - there is an item i with higher value than a chosen item j, but $x_i < w_i$, $x_j > 0$ and $v_i < v_j$
  - If we substitute some i with j, we get a better solution
  - How much of i: $\min\{w_i - x_i, x_j\}$
  - Thus, there is no better solution than the greedy one

Algorithm *fractionalKnapsack*(*S*, *W*)

  Input: set *S* of items w/ benefit b_i and weight w_i; max. weight *W*

  Output: amount x_i of each item *i* to maximize benefit w/ weight at most *W*

  for *each item i in S*

    $x_i \leftarrow 0$

    $v_i \leftarrow b_i / w_i$ {value}

  $w \leftarrow 0$ {total weight}

  while $w < W$

    remove item i w/ highest v_i

    $x_i \leftarrow \min\{w_i, W - w\}$

    $w \leftarrow w + \min\{w_i, W - w\}$

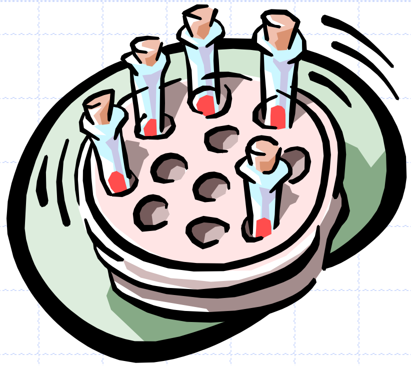
Analysis of Greedy Algorithm for Fractional Knapsack Problem

- We can sort the items by their benefit-to-weight values, and then process them in this order.
- This would require O(n log n) time to sort the items and then O(n) time to process them in the while-loop.
- To see that our algorithm is correct, suppose, for the sake of contradiction, that there is an optimal solution better than the one chosen by this greedy algorithm.
- Then there must be two items i and j such that

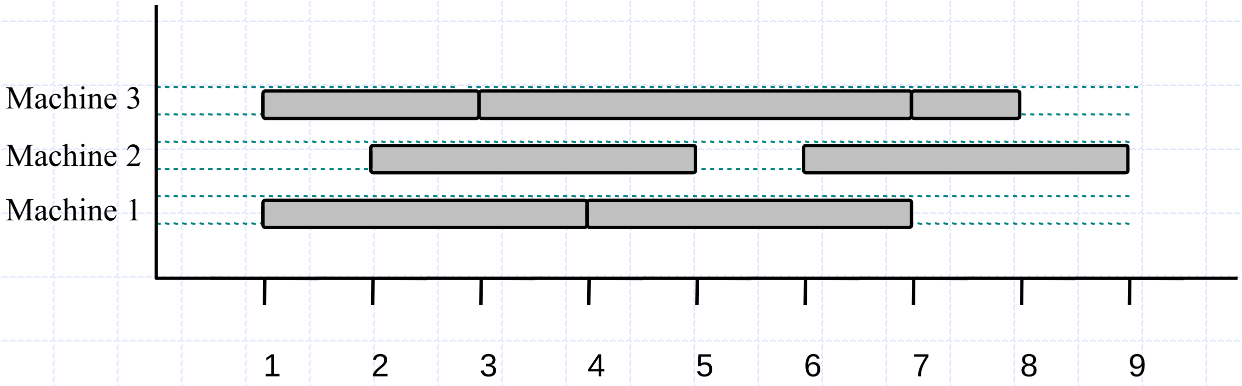
$$x_i < w_i,\ x_j > 0, \text{ and } v_i > v_j\ .$$

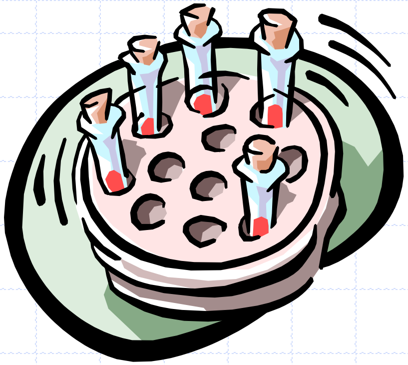
- Let $y = \min\{w_i - x_i, x_j\}$.
- But then we could replace an amount y of item j with an equal amount of item i, thus increasing the total benefit without changing the total weight, which contradicts the assumption that this non-greedy solution is optimal.

Task Scheduling



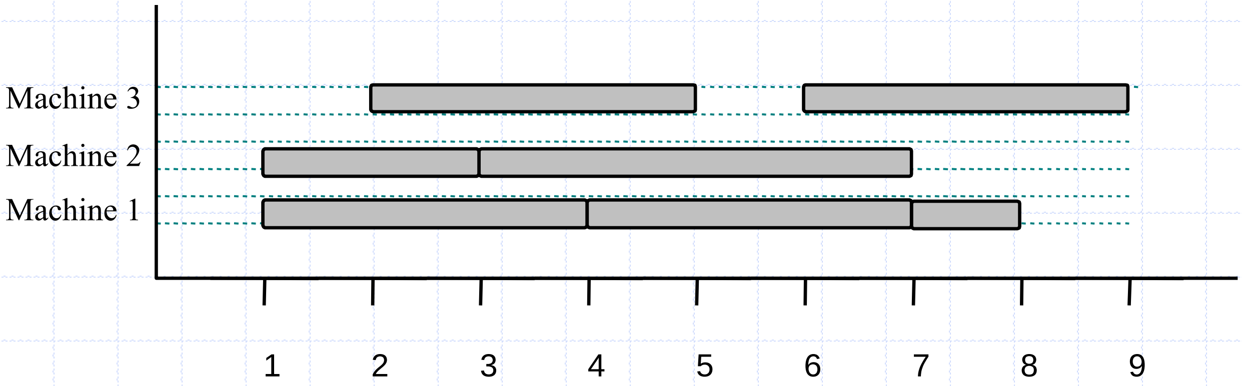
- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
- Goal: Perform all the tasks using a minimum number of "machines."



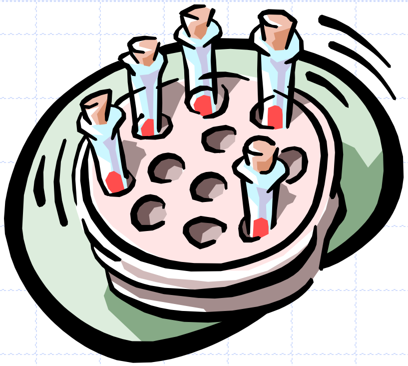


Example

- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
  - [1,4], [1,3], [2,5], [3,7], [4,7], [6,9], [7,8] (ordered by start)
- Goal: Perform all tasks on min. number of machines



Task Scheduling Algorithm



- Greedy choice: consider tasks by their start time and use as few machines as possible with this order.
  - Run time: O(n log n). Why?
- Correctness: Suppose there is a better schedule.
  - We can use k-1 machines
  - The algorithm uses k
  - Let i be first task scheduled on machine k
  - Task i must k-1 conflict with other tasks
  - But that means there is no non-conflicting schedule using k-1 machines

Algorithm *taskSchedule*(T)

Input: set T of tasks w/ start time s_i and finish time f_i

Output: non-conflicting schedule with minimum number of machines

$m \leftarrow 0$ {no. of machines}

while *T is not empty*

remove task i w/ smallest s_i

if *there's a machine j for i* **then**

schedule i on machine j

else

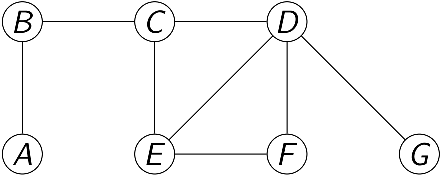
$m \leftarrow m + 1$

schedule i on machine m

Vertex Cover

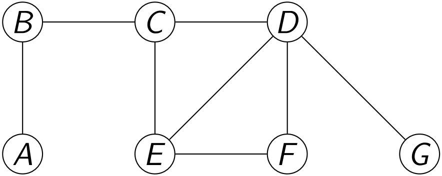
- Given a graph $G = (V, E)$.
- Select $V' \subseteq V$ such that
  - Every $(u, v) \in E$ has u or $v \in V'$
- Optimization: find smallest V'.

Example:



A Greedy Algorithm

- Idea: every edge gets represented



$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$ **do**
  Select any $e = (u, v) \in E'$
  Add u, v to C
  Remove all edges incident to u or v
return C

How accurate is it?

$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$ **do**
  Select any $e = (u, v) \in E'$
  Add u, v to C
  Remove all edges incident to u or v
return C